Back Paper

Time:	10:00–13:00, January 24, 2022.	Course name:	*Algebra I*
Degree:	MMath.	Year:	1^{st} Year, 1^{st} Semester; 2021–2022.
Course instructor:	Ramdin Mawia.	Total Marks:	50.

Attempt any three of the following problems, including problem nº 2. All rings are commutative with identity, and all ring morphisms take identity to identity.

Indian Statistical Institute Bangalore Centre

Rings and modules

1. Define and construct the tensor product of modules. State its universal property. **3+7+4+6 =20**

   (a) Define restriction and extension of scalars for modules. Let $A \to B$ be a ring morphism and let M be an A-module and N be a B-module. Show that there is a natural isomorphism of abelian groups

   $$\mathrm{Hom}_B(B \otimes_A M, N) \cong \mathrm{Hom}_A(M, N).$$

   Is it an isomorphism of A-modules? Justify.

   (b) What is the $\mathbb{C}$-vector space you obtain from the abelian group $(\mathbb{Z}/6\mathbb{Z}) \times \mathbb{Z} \times 2\mathbb{Z}$ by extending the ring of scalars from $\mathbb{Z}$ to $\mathbb{C}$? Justify your claim.

   (c) Let A be an integral domain with quotient field K and let B be a K-algebra. Let $M = K \otimes_A B$, so M is an A-algebra, and by extension of scalars, a K-algebra as well. Is it always true that

      i. $M \cong B$ considering both M and B as A-algebras?

      ii. $M \cong B$ considering both M and B as K-algebras?

   Give justifications.

2. Define a local ring. When do we say that a local ring is complete? Show that the ring $\mathbb{Z}_{(p)}$ of all rational numbers whose denominators (when written in their lowest forms) are not divisible by the given prime p, is a local ring. Is it complete? **10**

3. Let A be a ring. Define a short exact sequence of A-modules. When do we say that a short exact sequence is split? Let **3+8+9 =20**

   $$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

   be a short exact sequence of A-modules.

   (a) Let S be a multiplicative submonoid of A^*. Show that the sequence of A-modules

   $$0 \longrightarrow S^{-1}A \otimes_A L \xrightarrow{1\otimes f} S^{-1}A \otimes_A M \xrightarrow{1\otimes g} S^{-1}A \otimes_A N \longrightarrow 0$$

   is a short exact sequence. Here $1 \otimes f$ and $1 \otimes g$ are the A-linear morphisms induced by $(a/s, x) \mapsto (a/s) \otimes f(x)$ and $(a/s, x) \mapsto (a/s) \otimes g(x)$ respectively.

   (b) Suppose A is a PID and F is a torsion-free A-module. Show that for every short exact sequence

   $$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

   the induced sequence

   $$0 \longrightarrow F \otimes_A L \xrightarrow{1\otimes f} F \otimes_A M \xrightarrow{1\otimes g} F \otimes_A N \longrightarrow 0$$

   is also exact. Is it necessarily split when N is free?

4. Decide whether the following statements are true or false, with brief justifications (counterexamples, proofs, or such and such a theorem implies this etc) (**any ten**): **20**

   (a) The polynomial ring $\mathbb{Z}[X]$ is isomorphic to the power series ring $\mathbb{Z}[[X]]$.

   (b) Let A and B be torsion-free abelian groups such that $A \otimes_{\mathbb{Z}} \mathbb{Q} \cong B \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathbb{Q}$-vector spaces, then $A \cong B$ as abelian groups.

   (c) Let A be a UFD. A power series $a_0 + a_1X + \cdots \in A[[X]]$ is irreducible in $A[[X]]$ if a_0 is irreducible in A.

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(d) The power series ring $\mathbb{Z}/27\mathbb{Z}[[X]]$ is a complete local ring.

(e) For any ring morphism $A \to B$, we have $A[X] \otimes_A B \cong B[X]$ as A-modules.

(f) If A is a local ring, then $A[X]/\langle X^n \rangle$ is a local ring for each positive integer n.

(g) For any positive integers m and n, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$ with $d = \gcd(m, n)$.

(h) There is a $\mathbb{Z}$-module M such that the sequence $0 \to \mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow M \to 0$ is split short exact.

(i) In a short exact sequence of A-modules $0 \to M' \to M \to M'' \to 0$, if M' and M'' are finitely generated then so is M.

(j) If S is a multiplicative subset $\mathbb{Z}$ with $0 \notin S, 2 \in S$ then $S^{-1}\mathbb{Z}$ is a local ring.

(k) If I is an ideal of a Noetherian ring A, then A/I is a Noetherian ring.

(l) The polynomial $X^{2022} + 2X + 7$ is irreducible in $\mathbb{Z}[X]$.

(m) Every Noetherian local ring is complete.

(n) If A is a subring of $\mathbb{Q}[X]$ which strictly contains $\mathbb{Q}$ (i.e., $\mathbb{Q} \subsetneq A \subset \mathbb{Q}[X]$), then $\mathbb{Q}[X]$ is a finitely generated A-module.

(o) Let A be a ring and M be an A-module such that $M \otimes_A N \cong N$ for every A-module N, then M is a free A-module of rank 1.

(p) If I, J are comaximal ideals of a ring A, then $(A/I) \otimes_A (A/J)$ is the zero A-module.

(q) A finite direct sum of Noetherian modules is a Noetherian module.

(r) If M is a finitely generated A-module then $M^{\otimes n}$ is finitely generated for any $n \geqslant 1$.

(s) There is a $\mathbb{Q}$-vector space V such that the sequence $0 \to \mathbb{Q} \hookrightarrow \mathbb{R} \twoheadrightarrow V \to 0$ is split short exact.

(t) Any finitely generated $\mathbb{Z}$-algebra is isomorphic to a quotient of a polynomial ring over $\mathbb{Z}$.